Imprecise probabilities as a semantics for intuitive probabilistic reasoning

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March 12, 2019

Abstract

I prove a connection between the logical framework for intuitive probabilistic reasoning (IPR) introduced by Crane (2017) and sets of imprecise probabilities. More specifically, this connection provides a straightforward interpretation to sets of imprecise probabilities as subjective credal states, giving a formal semantics for Crane's formal system of IPR. The main theorem establishes the IPR framework as a potential logical foundation for imprecise probability that is independent of the traditional probability calculus.

Keywords and phrases: imprecise probability; credence; credal state; Choquet capacity; intuitionistic logic; Martin-Löf type theory

1 Introduction

In traditional subjectivist approaches to probability, degrees of belief are represented by a probability function (the Bayesian view (de Finetti, 1937)), as a non-additive belief function (as in the Dempster–Shafer theory (Shafer, 1976)), or more generally as a set of probabilities (in imprecise probability (Walley, 1990)). While the scope of imprecise probability extends beyond mere sets of probability, I focus here on the representation of subjective credal states by Choquet capacities of order 2, which provides a sufficiently general framework to illustrate the main result. In what follows, I shall refer to this paradigm of subjective belief generically as the *IP framework*.

Here I show a connection between the above IP framework and a new framework for subjective probability judgment, which I call *intuitive probabilistic reasoning* (IPR). The IPR formalism was introduced by Crane (2017), has been previously applied in a formal logical system for the notion of 'typicality' (Crane and Wilhelm, 2019), and is expanded in Crane (2019) as a logical framework for intuitive reasoning under uncertainty. The IPR formalism is independent of earlier frameworks of probability, introducing several new concepts, many of which lie beyond the scope of this brief note. Most germane to this article is its formal representation of subjective beliefs as mathematical objects other than conventional (imprecise) probability functions. I provide further technical preliminaries of this formalism in Section 2.

Within the IP paradigm above, subjective credences can be boiled down to one or more functions assigning to any claim a numerical degree of belief. In IPR, by contrast,

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probabilities are formalized topologically, most precisely as a homotopy type (i.e., a topological space up to homotopy equivalence) whose structure captures the relationships among different pieces of evidence for a claim. I demonstrate this distinction with the following example.

Within the Bayesian paradigm, an agent's degree of belief about a claim A reflects a disposition toward betting on the truth of A. That is, an agent claiming $P(A) = p$ is willing to pay up to \$$p$ to buy a contract that pays \$1 if A is true and \$0 if A is false; or would alternatively accept as little as \$$(1 - p)$ to sell a contract which requires the agent to pay out \$1 if A is true or \$0 if A is false. Within the more general IP framework, in which the agent's credence is given by a set of Choquet capacities $\mathcal{B}$, a betting quotient can be elicited by computing the lower and upper credences, respectively:

$$P_*(A) = \inf_{P \in \mathcal{B}} P(A) \quad \text{and}$$
$$P^*(A) = \sup_{P \in \mathcal{B}} P(A).$$

In the betting interpretation, the agent would be willing to pay up to $P_*(A)$ to buy a contract on A or accept payment as little as $P^*(A)$ to sell a contract on A, and would be unwilling to buy or sell for a price between $P_*(A)$ and $P^*(A)$.

In the IPR paradigm, an agent's belief about A is instead communicated as a judgment about evidence supporting the claim that 'A is probable'. The 'probability of A' in this setting is interpreted qualitatively, in the same sense in which the terms 'probable', 'likely', and such are invoked in everyday vernacular. More formally, the perspective of an individual agent is represented by $\mathbf{Bel}(A)$, a topological space (most precisely homotopy type) whose points correspond to pieces of evidence that would lead the agent to feel justified in making the judgment that 'A is probable', and whose topological structure (i.e., paths between points) represents the agent's judgment about the relationship between different pieces of evidence. The notation $\mathbf{Bel}(A)$ suggests the interpretation of a body of evidence for A, in the sense that the subject in possession of a piece of such evidence would feel justified in believing A. While possession of evidence may result in the same epistemic stance (namely belief in A), different justifications of belief $a : \mathbf{Bel}(A)$ and $a' : \mathbf{Bel}(A)$ constitute different beliefs, both formally and informally. Thus, instead of summarizing belief in A as $P(A) = p$ (a betting quotient), the agent reports a judgment of the form $a : \mathbf{Bel}(A)$, which is translated as 'A is probable on the basis of evidence a', or alternatively 'I believe A is probable because of a'.

More concretely, let A denote the claim that 'it is currently raining in New York City'. In the Bayesian paradigm, the agent who reports $P(A) = 0.20$ is willing to offer as high as 4-to-1 odds for a bet that loses if it is raining, or accept odds of 4-to-1 or greater for a bet that wins if it is raining. In the IPR paradigm, a concrete judgment of the form $a : \mathbf{Bel}(A)$ may be 'I believe it is raining in New York City because the weather report forecasted an 80% chance of rain today'. A different judgment of the same claim would be 'I believe it is raining because I saw a man carrying an umbrella'.

Crane (2017) proposed the latter as a logical framework for intuitive, probabilistic reasoning because it aims to formalize the process by which individuals reason about uncertain claims by justifying, explaining, or otherwise rationalizing their beliefs, not by quoting a betting quotient. In the above statement, the agent offers a *reason* for believing A, i.e., that the weather report forecasted an 80% chance of rain. This reason is not presented as proof that it is raining but rather as justification for why the agent feels justified in believing that it is probably raining. The agent who saw someone carrying an umbrella also holds the belief that it is raining, but for a different reason. In IPR, these reasons are the content of probability judgments.

As I have presented them here, the IP and IPR frameworks concern probabilistic judgments of a different nature. The former summarizes belief quantitatively in terms of a willingness to bet, while the latter expresses belief qualitatively by providing the reason that a given belief is justified. The key difference lies in the content which is communicated by the belief: in IP, the precise odds are stated but the reason for stating those odds is not; in IPR, the precise reason is given but the odds are not.

Either representation may be appropriate depending on the circumstances, and I will not discuss here which of these representations is preferred in any given context. I only note that statements of the latter type, which report a qualitative reason for belief without a precise numerical quantification, are widespread in everyday common sense reasoning as well as in justifying scientific hypotheses, mathematical conjectures, legal arguments, and the like. Rather than focus on the differences between these formalisms, I instead show a connection between them in which the syntax of IPR, which regards $\mathbf{Bel}(A)$ as a body of evidence formalized as a topological space, is interpreted semantically as a set of capacities, as in IP. This semantics suggests IPR as a potential logical foundation for imprecise probability that is autonomous from the traditional measure-theoretic foundation of probability.

Full comprehension of the IPR formalism requires considerable technical background in intuitionistic logic (Brouwer, 1981; Heyting, 1971; Dummett, 2000), Martin-Löf type theory (Martin-Löf, 1984, 1987, 1996), and homotopy type theory (Univalent Foundations Program, 2013; Tsementzis, 2017; Kapulkin and Lumsdaine, 2011), for which there is insufficient space in this brief note. For further technical details about IPR, the reader is referred to the above references and to the original article (Crane, 2017). Before showing the connection between IP and IPR, I give a brief introduction to necessary formal aspects of IPR in the next section.

2 Preliminaries

I restrict here to the minimal fragment of Martin-Löf type theory (MLTT) necessary to communicate the main idea behind IPR. Readers unfamiliar with type-theoretic notation can safely interpret the syntax set-theoretically, reading '$A : \mathbf{Type}$' as '$A \in \mathbf{Set}$' (A is a set) and '$a : A$' as '$a \in A$' (a is an element of A). More important than the formalism in type theory is the distinction between the claim itself (denoted A) and an assertion of knowledge or belief regarding the claim (denoted $a : A$ or $a : \mathbf{Bel}(A)$, respectively).

In the fragment of MLTT used here, there are two primitive statements, which are called *judgments* and which correspond to *types* and *terms*, respectively:

judgment	formal meaning	interpretation
$A : \mathbf{Type}$	A is a Type	A is a claim
$a : A$	a is a term of type A	a is evidence for A

In the example of the previous section, A is the claim 'it is raining in New York City' and a term $a : A$ is a reason for believing A, e.g., 'because I see through the window that it is raining'. This interpretation of types as claims and terms as proofs has a long history at the intersection of logic and type theory and is called the Curry–Howard *propositions-as-types* interpretation (Curry and Feys 1959; Howard, 1969).

In the IPR framework, we expand upon the Curry–Howard interpretation by associating to each claim A, whose terms are evidence that A is true, a claim $\mathbf{Bel}(A)$, whose terms are evidence that A is probable. So, in our running rain example, $a : A$ is a justification for believing that A is true whereas $a' : \mathbf{Bel}(A)$ is a justification for believing that A is probable or likely (e.g., because the weatherman forecasted an 80% chance of rain).

Finally, by the intuitionistic nature of MLTT, it is intended that judgments are interpreted relative to the context in which the agent makes it. Clearly, an agent who has knowledge of the 80% weather forecast will be in a different frame of mind with respect to the claim of rain than someone without knowledge of the forecast, and so the logic ought to reflect this difference in perspective. When expressing logical deductions formally, we write capital Greek letters $\Delta, \Gamma, \Xi, \ldots$ to denote a generic *context* in which a judgment is being made. Altogether, every judgment in MLTT is expressed in the general form:

$$\text{Context } \Delta \quad \vdash \quad \text{Judgment } \mathcal{J},$$

which is interpreted to mean 'Judgment $\mathcal{J}$ is justified in Context Δ', where the judgment has the form of one of the two primitive judgments written above.

3 Axioms of justified belief

Crane (2017) suggests two core axioms of intuitive probabilistic reasoning about evidence:

(i) Truth implies probability: for any claim A, $A \to \mathbf{Bel}(A)$.

(ii) If A implies B then A is probable implies B is probable: for any claims A and B,

$$(A \to B) \to (\mathbf{Bel}(A) \to \mathbf{Bel}(B)).$$

In the type-theoretic framework, the justification (i.e., the term) is a substantive component of any judgment, making the logical implications above insufficient for expressing these axioms. Believing that it is raining on the basis of a weather forecast is different than believing on the basis of seeing a man carrying an umbrella. To this end, the treatment of logical implication $A \to B$ marks one the main differences between classical logic and intuitionistic type theory. Classically, the implication $A \to B$ is the material conditional $\neg A \vee B$, read as *if A then B*. In the constructive logic of type theory, however, the implication $A \to B$ requires that a witness for B can be explicitly constructed from any witness of A. Thus, in MLTT, logical implication $A \to B$ is formally a function $f : A \to B$ with domain A and codomain B which can be applied to any justification for A (i.e., $a : A$) in order to obtain a justification for B (i.e., $f(a) : B$).

The formal axiomatization of IPR (with **Type** written as **Claim** for emphasis) is expressed by the following three logical rules:

Formation rule:

$$\frac{\Delta \vdash A : \mathbf{Claim}}{\Delta \vdash \mathbf{Bel}(A) : \mathbf{Claim}} \quad (\mathbf{Bel}\text{-form}) \tag{1}$$

Semi-formal: If 'A is true' is a valid claim in context Δ, then 'A is probable' is a valid claim in Δ.

In the above semi-formal explanation, the description of A as being 'a valid claim' in context Δ reflects Martin-Löf's intuitionistic conception of meaning:

> "the meaning of a proposition [...] is determined by that which counts as a verification of it." (Martin-Löf, 1996, p. 27)

With this perspective, it is implicit to the assertion that 'A is a claim' that the claim has meaning, in the sense that the subject making the assertion knows what counts as a verification of the claim. It is in this sense that a claim is considered valid. I will not belabor this point any further here; see Martin-Löf (1996) for further discussion.

Introduction rule:

$$\frac{\Delta \vdash A : \mathbf{Claim}}{\Delta, a : A \vdash \mathtt{evid}_A(a) : \mathbf{Bel}(A)} \quad (\mathbf{Bel}\text{-intro}) \tag{2}$$

Semi-formal: Evidence that A is true provides evidence that A is probable.

Elimination rule:

$$\frac{\begin{array}{c}\Delta \vdash A : \mathbf{Claim} \\ \Delta \vdash C : \mathbf{Claim} \\ \Delta, a : A \vdash d(a) : C\end{array}}{\Delta, x : \mathbf{Bel}(A) \vdash \mathtt{imp}_d(x) : \mathbf{Bel}(C)} \quad (\mathbf{Bel}\text{-elim}) \tag{3}$$

Semi-formal: If evidence for A provides evidence for C in context Δ, then evidence for the probability of A implies evidence for the probability of C in Δ.

In some applications, it may also be appropriate to add the following axiom about belief in the logical contradiction, represented by the type $\mathbf{0} : \mathbf{Type}$ in MLTT.

0-rule:

$$\frac{\Delta \ \mathbf{ctx}}{\Delta, x : \mathbf{Bel}(\mathbf{0}) \vdash \sigma(x) : \mathbf{0}} \quad (\mathbf{Bel}\text{-}\mathbf{0}) \tag{4}$$

Semi-formal: Belief in a vacuous claim is not justified in any context.

More precisely, the **0**-rule says that any evidence that a contradiction is probable provides evidence for the contradiction itself.

From these axioms alone it is possible to derive a number of intuitive results about the relationship between beliefs about individual claims and beliefs about their conjunction, disjunction, and negation. These results are deferred to Crane (2017). Here we focus on a specific interpretation of these axioms in terms of imprecise probabilities.

4 Connection to imprecise probabilities

For a set Ω, we let $\text{Choquet}_2(\Omega)$ denote the set of all *Choquet capacities of order 2* with base set Ω. In particular, the elements of $\text{Choquet}_2(\Omega)$ are triples $(\Omega, \mathcal{F}, P_*)$, where $\mathcal{F}$ is a σ-algebra on Ω and P_* is a set function $\mathcal{F} \to [0, 1]$ satisfying

(I) $P_*(\Omega) = 1$ and $P_*(\emptyset) = 0$,

(II) $P_*(A) \geq 0$ for all $A \in \mathcal{F}$, and

(III) $P_*(A \cup B) \geq P_*(A) + P_*(B) - P_*(A \cap B)$ for all $A, B \in \mathcal{F}$.

We interpret Ω as a universe of possible states of the world, so that each claim A about the universe is represented by a subset $\tilde{A} \subseteq \Omega$ consisting of all worlds at which A holds. A particular $(\Omega, \mathcal{F}, P_*) \in \text{Choquet}_2(\Omega)$ can be interpreted as a subjective disposition toward the states of universe Ω, with $\tilde{A} \in \mathcal{F}$ indicating that the subject has a disposition about A (i.e., assigns a credence to $\tilde{A}$) and $P_*(\tilde{A})$ recording the credence. The function P_* represents credence from a particular frame of reference. In general, we define a *credal state* to be a subset of capacities $\mathcal{S} \subseteq \text{Choquet}_2(\Omega)$. Singleton sets thus correspond to credal states represented by a unique capacity P_*, but we also allow credal states that are sets of such functions in order to reason more generally about uncertain propositions without having to know the precise credence about all states of the world.

In connecting credal states to IPR, a credal state $\mathcal{S}$ for which $\tilde{A} \in \mathcal{F}$ for every $(\Omega, \mathcal{F}, P_*) \in \mathcal{S}$ is one in which A is a valid claim, in the sense that an agent with any one of the dispositions in $\mathcal{S}$ assigns credence to A, and $\mathcal{S}$ for which $P_*(A) = 1$ for every $(\Omega, \mathcal{F}, P_*) \in \mathcal{S}$ is a credal state representing belief that 'A is true'. We extend this setup to include beliefs about the probability of A by adding a set $\widetilde{\mathbf{Bel}(A)}$ to $\mathcal{F}$ for each $\tilde{A} \in \mathcal{F}$ and specifying a threshold $1/2 < t \leq 1$ such that $P_*(\widetilde{\mathbf{Bel}(A)}) = 1$ whenever $P_*(\tilde{A}) \geq t$. A credal state $\mathcal{S}$ for which $P_*(A) \geq t$ for every $(\Omega, \mathcal{F}, P_*) \in \mathcal{S}$ thus represents the belief that 'A is probable', in the sense that an agent possessing any credence in $\mathcal{S}$ believes that A is sufficiently probable.

We connect the semantics of IP with the syntax of IPR as follows. First, we write Δ **ctx** to denote that Δ is a well-formed context according to the rules of MLTT. (For a full list of rules of MLTT, see either Univalent Foundations Program (2013) or the appendix of Kapulkin and Lumsdaine (2011).) For any $\tilde{A} \subseteq \Omega$, we define

$$\begin{aligned}
\mathcal{F}_A &:= \{(\Omega, \mathcal{F}, P_*) \in \text{Choquet}_2(\Omega) \mid \tilde{A} \in \mathcal{F}\} \\
P_A &:= \{(\Omega, \mathcal{F}, P_*) \in \text{Choquet}_2(\Omega) \mid \tilde{A} \in \mathcal{F} \quad \text{and} \\
&\qquad P_*(\tilde{A}) = 1\} \\
P_{\mathbf{Bel}(A)} &:= \{(\Omega, \mathcal{F}, P_*) \in \text{Choquet}_2(\Omega) \mid \tilde{A} \in \mathcal{F} \quad \text{and} \\
&\qquad P_*(\tilde{A}) \geq t\},
\end{aligned}$$

for fixed $1/2 < t \leq 1$. When translating the syntax of IPR into the semantics of IP, we interpret the turnstile '$\vdash$' as '$\subseteq$' and commas on the left side of the turnstile as $\cap$. With this translation, the basic judgments of MLTT are interpreted as:

Syntax (IPR)	Semantics (IP)
Δ **ctx**	$\Delta \subseteq \text{Choquet}_2(\Omega)$
$\Delta \vdash A : \mathbf{Type}$	$\Delta \subseteq \mathcal{F}_A$
$\Delta \vdash a : A$	$\Delta \subseteq P_A$
$\Delta \vdash a' : \mathbf{Bel}(A)$	$\Delta \subseteq P_{\mathbf{Bel}(A)}$.

By this correspondence,

- a well-formed context Δ **ctx** in IPR corresponds to a credal state $\Delta \subseteq \text{Choquet}_2(\Omega)$. In the former, Δ is the perspective from which an agent makes a judgment; and in the latter, this perspective is represented by a set of capacities representing the agent's possible credences;

- a judgment $\Delta \vdash A : \mathbf{Claim}$ in IPR is interpreted as 'A is a valid claim in context Δ', which in IP corresponds to the credal state $\Delta \subseteq \mathcal{F}_A$ in which every member assigns credence to A;
- the assertion $\Delta \vdash a : A$ in IPR is interpreted as 'a is evidence for A in context Δ', which in IP corresponds to the credal state $\Delta \subseteq P_A$ in which every member assigns credence 1 to A; and
- the assertion $a' : \mathbf{Bel}(A)$ in IPR is interpreted as 'a' is evidence that A is probable in context Δ', which in IP corresponds to a credal state $\Delta \subseteq P_{\mathbf{Bel}(A)}$ in which every member assigns sufficiently high credence to A.

To establish IP as a semantics for IPR, I prove soundness of the above translation in terms of the rules for MLTT and the type **Bel** defined above. The syntax of MLTT has additional type formers $\times$, $+$, and $\mathbf{0}$ corresponding to conjunction $\wedge$, disjunction $\vee$, and the contradiction $\perp$, respectively. In particular, for $A, B : \mathbf{Claim}$, $A \times B : \mathbf{Claim}$ is the claim 'A and B', $A + B : \mathbf{Claim}$ is 'A or B', and $\mathbf{0} : \mathbf{Claim}$ is the vacuous claim. The full translation from MLTT into set theory is given in the table below.

MLTT	set theory
Δ **ctx**	$\Delta \subseteq \text{Choquet}_2(\Omega)$
$A : \mathbf{Type}$	$\mathcal{F}_A$
$a : A$	P_A
$\vdash$	$\subseteq$
$A \times B$	$\tilde{A} \cap \tilde{B}$
$A + B$	$\tilde{A} \cup \tilde{B}$
$\mathbf{0}$	$\emptyset$

Theorem 1. *IP semantics is sound for IPR.*

To prove soundness, we interpret each of the rules for the $\times$, $+$, $\mathbf{0}$, and **Bel** types in IPR into the semantics of sets of Choquet capacities and show that the rule holds. We begin by specifying the interpretation of the rules for contexts. In the following displays, the lefthand side shows the rules of MLTT (see Univalent Foundations Program (2013)) and the righthand side is the translation into IP according to the above protocol.

- **Structural rules**, **•-ctx**:

Syntax	Semantics
$\dfrac{}{\bullet\ \mathbf{ctx}}$	$\dfrac{}{\text{Choquet}_2(\Omega) \subseteq \text{Choquet}_2(\Omega)}$

Holds trivially: Every set is a subset of itself.[1]

[1]This rule states that there is an initial 'empty' context •. In the semantics, the context places constraints on an agent's credal states, and thus this initial 'empty' context corresponds to a context without constraints, i.e., $\Delta \equiv \text{Choquet}_2(\Omega)$.

- **Structural rules**, ext-**ctx**

$$\text{Syntax} \quad \frac{\Delta\ \textbf{ctx} \qquad \Delta \vdash A : \textbf{Claim}}{\Delta, x : A\ \textbf{ctx}} \qquad\qquad \text{Semantics} \quad \frac{\Delta \subseteq \text{Choquet}_2(\Omega) \qquad \Delta \subseteq \mathcal{F}_A}{\Delta \cap P_A \subseteq \text{Choquet}_2(\Omega)}$$

  By assumption $\Delta \subseteq \text{Choquet}_2(\Omega)$, and thus $\Delta \cap S \subseteq \Delta \subseteq \text{Choquet}_2(\Omega)$ for all other sets A. Instantiating $S = P_A$ gives the result.

- **Structural rules**, ax-**ctx**

$$\text{Syntax} \quad \frac{\Delta, a : A, \Xi\ \textbf{ctx}}{\Delta, a : A, \Xi \vdash a : A} \qquad\qquad \text{Semantics} \quad \frac{\Delta \cap P_A \cap \Xi \subseteq \text{Choquet}_2(\Omega)}{\Delta \cap P_A \cap \Xi \subseteq P_A}$$

  By assumption, $\Delta \cap P_A \cap \Xi$ is a set, and for any sets S and T it is always the case that $S \cap T \subseteq S$, yielding the result.

It follows from these structural rules for contexts that every context is a finite list of judgments of the form

$$(a_1 : A_1, \ldots, a_n : A_n)\ \textbf{ctx},$$

which in our semantic interpretation translates to

$$P_{A_1} \cap \cdots \cap P_{A_n} \subseteq \text{Choquet}_2(\Omega).$$

Thus, in our semantic treatment, every context Δ can be expressed in the form

$$\Delta \equiv P_{A_1} \cap \cdots \cap P_{A_n} \tag{5}$$

for some finite list $A_1, \ldots, A_n \subseteq \Omega$. This specific representation will become useful when we prove soundness for the coproduct and **Bel**-types below.

We next prove soundness for the product type.

- **Product type**, formation rule:

$$\text{Syntax} \quad \frac{\Delta \vdash A : \textbf{Claim} \qquad \Delta \vdash B : \textbf{Claim}}{\Delta \vdash A \times B : \textbf{Claim}} \qquad\qquad \text{Semantics} \quad \frac{\Delta \subseteq \mathcal{F}_A \qquad \Delta \subseteq \mathcal{F}_B}{\Delta \subseteq \mathcal{F}_{A \times B}}$$

  Let $(\Omega, \mathcal{F}, P_*) \in \Delta$ so that $\tilde{A}, \tilde{B} \in \mathcal{F}$. Then $\widetilde{A \times B} \equiv \tilde{A} \cap \tilde{B} \in \mathcal{F}$ because $\mathcal{F}$ is an algebra on Ω and thus is closed under intersection. It follows that $\Delta \subseteq \mathcal{F}_{A \times B}$, as claimed.

- **Product type**, introduction rule:

$$\frac{\begin{array}{c}\text{Syntax}\\ \Delta \vdash A : \textbf{Claim} \\ \Delta \vdash B : \textbf{Claim}\end{array}}{\Delta, a : A, b : B \vdash (a,b) : A \times B} \qquad \frac{\begin{array}{c}\text{Semantics}\\ \Delta \subseteq \mathcal{F}_A \\ \Delta \subseteq \mathcal{F}_B\end{array}}{\Delta \cap P_A \cap P_B \subseteq P_{A\times B}}$$

By the formation rule, $\Delta \subseteq \mathcal{F}_A$ and $\Delta \subseteq \mathcal{F}_B$ implies $\Delta \subseteq \mathcal{F}_{A\times B}$, so that any $(\Omega, \mathcal{F}, P_*) \in \Delta$ has $\tilde{A} \cap \tilde{B} \in \mathcal{F}$ because $\mathcal{F}$ is an algebra. Furthermore, assume $(\Omega, \mathcal{F}, P_*) \in \Delta \cap P_A \cap P_B$ so that by axiom (III) of Choquet capacities, we have

$$\begin{aligned} P_*(\tilde{A} \cap \tilde{B}) &\geq P_*(\tilde{A}) + P_*(\tilde{B}) - P_*(\tilde{A} \cup \tilde{B}) \\ &= 1 + 1 - P_*(\tilde{A} \cup \tilde{B}). \end{aligned}$$

Finally, since $0 \leq P_*(S) \leq 1$ for all $S \in \mathcal{F}$, we must have $P_*(\tilde{A} \cup \tilde{B}) \leq 1$, implying $P_*(\tilde{A} \cap \tilde{B}) \geq 1$, and thus $P_*(\tilde{A} \cap \tilde{B}) = 1$, as claimed.

- **Product type**, elimination rule:

$$\frac{\begin{array}{c}\text{Syntax}\\ \Delta \vdash A : \textbf{Claim} \\ \Delta \vdash B : \textbf{Claim} \\ \Delta, a : A, b : B \vdash C : \textbf{Claim} \\ \Delta, a : A, b : B \vdash d(a,b) : C\end{array}}{\Delta, z : A \times B \vdash \texttt{split}_d(z) : C} \qquad \frac{\begin{array}{c}\text{Semantics}\\ \Delta \subseteq \mathcal{F}_A \\ \Delta \subseteq \mathcal{F}_B \\ \Delta \cap \mathcal{F}_A \cap \mathcal{F}_B \subseteq \mathcal{F}_C \\ \Delta \cap P_A \cap P_B \subseteq P_C\end{array}}{\Delta \cap P_{A\times B} \subseteq P_C}$$

For $(\Omega, \mathcal{F}, P_*) \in \Delta \cap P_{A\times B} \subseteq P_{A\times B}$, we have

$$\min(P_*(\tilde{A}), P_*(\tilde{B})) \geq P_*(\tilde{A} \cap \tilde{B}) = 1;$$

whence $P_*(\tilde{A}) = P_*(\tilde{B}) = 1$ and $(\Omega, \mathcal{F}, P_*) \in \Delta \cap P_A \cap P_B$. By assumption, we have $\Delta \cap P_A \cap P_B \subseteq P_C$ so that $(\Omega, \mathcal{F}, P_*) \in P_C$, and thus $\Delta \cap P_{A\times B} \subseteq P_C$, as claimed.

Before we move on to discuss the coproduct type, we can use the rules for product type to deduce that $P_A \cap P_B = P_{A\times B}$ for all A, B : **Claim**. From this and the representation of contexts in the form (5), we can equivalently express any context as

$$\Delta \equiv P_{A_1 \times \cdots \times A_n}, \tag{6}$$

which can more compactly be written as

$$\Delta \equiv P_\Phi$$

for some $\tilde{\Phi} \in \mathcal{F}$, because $\tilde{A}_1 \cap \cdots \cap \tilde{A}_n \in \mathcal{F}$ whenever $\tilde{A}_1, \ldots, \tilde{A}_n \in \mathcal{F}$. This representation plays a role in our proof of soundness for the coproduct elimination rule.

- **Coproduct type**, formation rule:

$$\frac{\begin{array}{c}\text{Syntax}\\ \Delta \vdash A : \textbf{Claim} \\ \Delta \vdash B : \textbf{Claim}\end{array}}{\Delta \vdash A + B : \textbf{Claim}} \qquad \frac{\begin{array}{c}\text{Semantics}\\ \Delta \subseteq \mathcal{F}_A \\ \Delta \subseteq \mathcal{F}_B\end{array}}{\Delta \subseteq \mathcal{F}_{A+B}}$$

Let $(\Omega, \mathcal{F}, P_*) \in \mathcal{F}_A \cap \mathcal{F}_B$, then $\tilde{A}^c \in \mathcal{F}$, $\tilde{B}^c \in \mathcal{F}$, and $\tilde{A} \cap \tilde{B} \in \mathcal{F}$, because $\mathcal{F}$ is an algebra. Finally, by definition we have $\widetilde{A+B} \equiv \tilde{A} \cup \tilde{B} \equiv (\tilde{A}^c \cap \tilde{B}^c)^c \in \mathcal{F}$, because $\mathcal{F}$ is a σ-algebra and is closed under complementation and intersection. It follows that $(\Omega, \mathcal{F}, P_*) \in \mathcal{F}_{A+B}$.

- **Coproduct type**, left introduction rule:

Syntax	Semantics
$\dfrac{\Delta \vdash A : \textbf{Claim} \quad \Delta \vdash B : \textbf{Claim}}{\Delta, a : A \vdash \texttt{inl}(a) : A + B}$	$\dfrac{\Delta \subseteq \mathcal{F}_A \quad \Delta \subseteq \mathcal{F}_B}{\Delta \cap P_A \subseteq P_{A+B}}$

By assumption, $\Delta \subseteq \mathcal{F}_A \cap \mathcal{F}_B$ implies that any $(\Omega, \mathcal{F}, P_*) \in \Delta$ assigns credence to A and B. The final premise $\Delta \cap P_A$ implies that $P_*(\tilde{A}) = 1$. By the preceding formation rule, we have $\tilde{A} \cup \tilde{B} \in \mathcal{F}$, and so P_* assigns credence to it, and since P_* is increasing we must have $P_*(\tilde{A} \cup \tilde{B}) \geq P_*(\tilde{A}) = 1$; whence $P_* \in P_{A+B}$, as claimed.

The same argument carries through to prove the right introduction rule for the coproduct type.

- **Coproduct type**, elimination rule:

Syntax	Semantics
$\dfrac{\Delta \vdash A : \textbf{Claim} \quad \Delta \vdash B : \textbf{Claim} \quad \Delta, a : A, b : B \vdash C : \textbf{Claim} \quad \Delta, a : A \vdash d_l(a) : C \quad \Delta, b : B \vdash d_r(b) : C}{\Delta, z : A + B \vdash \texttt{case}_{d_l, d_r}(z) : C}$	$\dfrac{\Delta \subseteq \mathcal{F}_A \quad \Delta \subseteq \mathcal{F}_B \quad \Delta \cap P_A \cap P_B \subseteq \mathcal{F}_C \quad \Delta \cap P_A \subseteq P_C \quad \Delta \cap P_B \subseteq P_C}{\Delta \cap P_{A+B} \subseteq P_C}$

Here we use the representation in (6) to express $\Delta \equiv P_\Phi$ for some $\Phi \equiv A_1 \times \cdots \times A_n$, so that the third and fourth assumptions and the conclusion on the righthand side, respectively, become

$$P_{\Phi \times A} \subseteq P_C, \quad P_{\Phi \times B} \subseteq P_C \quad \text{and} \quad P_{\Phi \times (A+B)} \subseteq P_C.$$

By assumption, we have $P_{\Phi \times A} \subseteq P_C$. Thus, any P_* that satisfies $P_*(\tilde{\Phi} \cap \tilde{A}) = 1$ must also satisfy $P_*(\tilde{C}) = 1$, which is possible only if $\tilde{\Phi} \cap \tilde{A} \subseteq \tilde{C}$. For suppose that there is some $\omega \in \tilde{\Phi} \cap \tilde{A}$ for which $\omega \notin \tilde{C}$. Then there is a measurable space $(\Omega, \mathcal{F}_\omega, P_\omega)$ with σ-algebra $\mathcal{F}_\omega = \{\Omega, \emptyset, \{\omega\}, \Omega \setminus \{\omega\}\}$ and P_ω the atomic measure at $\{\omega\}$ (i.e., $P_\omega(\{\omega\}) = 1$). With $\omega \in \tilde{\Phi} \cap \tilde{A}$, it follows that $P_\omega(\tilde{\Phi} \cap \tilde{A}) \geq P_\omega(\{\omega\}) = 1$ and $P_\omega(\tilde{C}) = 0$, contradicting the assumption. By applying an analogous argument to the fourth assumption, we must have $\tilde{\Phi} \cap \tilde{A} \subseteq \tilde{C}$.

Finally, note that $\tilde{\Phi} \cap (\tilde{A} \cup \tilde{B}) \equiv (\tilde{\Phi} \cap \tilde{A}) \cup (\tilde{\Phi} \cap \tilde{B})$, so that the conclusion reads

$$P_{(\Phi \times A)+(\Phi \times B)} \subseteq P_C.$$

Now, suppose $(\Omega, \mathcal{F}, P_*) \in P_{(\Phi \times A)+(\Phi \times B)}$ so that $P_*((\tilde{\Phi} \cap \tilde{A}) \cup (\tilde{\Phi} \cap \tilde{B})) = 1$. Then by the preceding argument we have $\tilde{\Phi} \cap \tilde{A} \subseteq \tilde{C}$ and $\tilde{\Phi} \cap \tilde{B} \subseteq \tilde{C}$, which implies

$$(\tilde{\Phi} \cap \tilde{A}) \cup (\tilde{\Phi} \cap \tilde{B}) \subseteq \tilde{C}.$$

It follows that

$$1 = P_*((\tilde{\Phi} \cap \tilde{A}) \cup (\tilde{\Phi} \cap \tilde{B})) \leq P_*(\tilde{C});$$

whence, $P_*(\tilde{C}) = 1$ and $(\Omega, \mathcal{F}, P_*) \in P_C$, as claimed.

We next discuss the **0** type.

- **0-type**, formation rule:

$$\begin{array}{cc} \text{Syntax} & \text{Semantics} \\ \dfrac{\Delta\ \mathbf{ctx}}{\Delta \vdash \mathbf{0} : \mathbf{Claim}} & \dfrac{\Delta \subseteq \text{Choquet}_2(\Omega)}{\Delta \subseteq \mathcal{F}_{\mathbf{0}}} \end{array}$$

As any σ-algebra contains $\tilde{\mathbf{0}} = \emptyset$ it is immediate that $\text{Choquet}_2(\Omega) = \mathcal{F}_{\mathbf{0}} = \{(\Omega, \mathcal{F}, P_*) \in \text{Choquet}_2(\Omega) \mid \emptyset \in \mathcal{F}\}$ and the conclusion follows.

- **0 type**, elimination rule:[2]

$$\begin{array}{cc} \text{Syntax} & \text{Semantics} \\ \dfrac{\Delta \vdash A : \mathbf{Claim}}{\Delta, x : \mathbf{0} \vdash \mathtt{efq}_A(x) : A} & \dfrac{\Delta \subseteq \mathcal{F}_A}{\Delta \cap P_{\mathbf{0}} \subseteq P_A} \end{array}$$

The subset $P_{\mathbf{0}} \subseteq \text{Choquet}_2(\Omega)$ consists of all $(\Omega, \mathcal{F}, P_*)$ that assign credence 1 to $\emptyset$. By axiom (I) of Choquet capacities, $P_*(\emptyset) = 0$, and thus $P_{\mathbf{0}} = \emptyset$ and $\Delta \cap P_{\mathbf{0}} \equiv \emptyset \subseteq P_A$ holds trivially.

Finally, for the **Bel**-type.

- **Belief type**, formation rule:

$$\begin{array}{cc} \text{Syntax} & \text{Semantics} \\ \dfrac{\Delta \vdash A : \mathbf{Claim}}{\Delta \vdash \mathbf{Bel}(A) : \mathbf{Claim}} & \dfrac{\Delta \subseteq \mathcal{F}_A}{\Delta \subseteq \mathcal{F}_{\mathbf{Bel}(A)}} \end{array}$$

We require that $\widetilde{\mathbf{Bel}(A)}$ is a measurable set whenever $\tilde{A}$ is, so that the conclusion immediately follows by our extended definition of $\text{Choquet}_2(\Omega)$.

- **Belief type**, introduction rule:

$$\begin{array}{cc} \text{Syntax} & \text{Semantics} \\ \dfrac{\Delta \vdash A : \mathbf{Claim}}{\Delta, a : A \vdash \mathtt{evid}_A(a) : \mathbf{Bel}(A)} & \dfrac{\Delta \subseteq \mathcal{F}_A}{\Delta \cap P_A \subseteq P_{\mathbf{Bel}(A)}} \end{array}$$

[2] Here `efq` stands for *ex falso quodlibet* ("from falsehood, anything follows"). Formally, this rule says that given any A : **Claim** and a proof $x : \mathbf{0}$ of the contradiction it is possible to construct a proof $\mathtt{efq}_A(x) : A$.

As any $(\Omega, \mathcal{F}, P_*) \in \Delta \cap P_A$ must satisfy $P_*(A) = 1$ and it is required that $\widetilde{\mathbf{Bel}(A)} \supseteq \tilde{A}$, we must have $P_*(\widetilde{\mathbf{Bel}(A)}) \geq P_*(\tilde{A}) = 1$, and $(\Omega, \mathcal{F}, P_*) \in P_{\mathbf{Bel}(A)}$, as claimed.

- **Belief type**, elimination rule:

Syntax	Semantics
$\Delta \vdash A : \mathbf{Claim}$	$\Delta \subseteq \mathcal{F}_A$
$\Delta \vdash B : \mathbf{Claim}$	$\Delta \subseteq \mathcal{F}_B$
$\Delta, a : A \vdash d(a) : B$	$\Delta \cap P_A \subseteq P_B$
$\Delta, x : \mathbf{Bel}(A) \vdash \mathtt{imp}_d(x) : \mathbf{Bel}(B)$	$\Delta \cap P_{\mathbf{Bel}(A)} \subseteq P_{\mathbf{Bel}(B)}$

By (6), we can express the second assumption as $P_{\Phi \cap A} \subseteq P_B$ for some $\tilde{\Phi} \subseteq \Omega$, from which it follows that $\tilde{\Phi} \cap \tilde{A} \subseteq \tilde{B}$ by an argument already given above when proving the elimination rule for the coproduct type. Thus, we can rewrite the conclusion as

$$P_{\Phi \times \mathbf{Bel}(A)} \subseteq P_{\mathbf{Bel}(B)}.$$

Let $(\Omega, \mathcal{F}, P_*) \in P_{\Phi \times \mathbf{Bel}(A)}$. Then $P_*(\tilde{\Phi} \cap \widetilde{\mathbf{Bel}(A)}) = 1$, and in particular $P_*(\tilde{\Phi}) = 1$ and $P_*(\widetilde{\mathbf{Bel}(A)}) = 1$, implying that $P_*(\tilde{A}) \geq 1 - t$.

By definition we have

$$\begin{aligned} P_*(\tilde{\Phi} \cap \tilde{A}) &\geq P_*(\tilde{\Phi}) + P_*(\tilde{A}) - P_*(\tilde{\Phi} \cup \tilde{A}) \\ &\geq P_*(\tilde{\Phi}) + (1 - t) - 1 \\ &= 1 - t. \end{aligned}$$

By assumption, we have $P_*(\tilde{B}) \geq P_*(\tilde{\Phi} \cap \tilde{A}) \geq 1 - t$, and $P_*(\widetilde{\mathbf{Bel}(B)}) = 1$ by definition.

- **Belief type**, **0** rule:

Syntax	Semantics
Δ **ctx**	$\Delta \subseteq \text{Choquet}_2(\Omega)$
$\Delta, x : \mathbf{Bel}(\mathbf{0}) \vdash \sigma(x) : \mathbf{0}$	$\Delta \cap P_{\mathbf{Bel}(\mathbf{0})} \subseteq P_{\mathbf{0}}$

By axiom (I), every Choquet capacity assigns 0 credence to $\tilde{\mathbf{0}} \equiv \emptyset$. Thus there does not exist any P_* for which $P_*(\emptyset) \geq 1 - t$, and it follows that $\Delta \cap P_{\mathbf{Bel}(\mathbf{0})} \subseteq P_{\mathbf{0}} \equiv \emptyset$, as required.

5 Concluding Remarks

The above proof of soundness establishes a formal connection between the logical formalism for justified belief introduced in Crane (2017) and the representation of credal states as sets of imprecise probabilities. In this semantics, we fix a universe Ω and represent a credal state as a subset of Choquet capacities of order 2 on Ω, of which Dempster–Shafer belief functions and probability functions are special cases. In light of this connection, it is worthwhile to explore how the IPR framework might be useful for studying general properties about models in IP. In particular, IPR could be explored as a formal foundation of imprecise probability, in the sense that many formal statements in IP can be interpreted into IPR. The existence of such a foundation offers at least two potential benefits:

1. It provides a standard of rigor which allows general theorems in IP to be established by abstracting to IPR and proving a number of special cases all at once. The amenability of MLTT, and thus IPR, to computerized proof assistants, such as Coq, offers another potential practical benefit when attempting such proofs.

2. IPR as a foundation for IP would establish imprecise probability as autonomous from classical probability theory, making precise the concept of what is an imprecise probability, without the need to refer to traditional 'precise' probabilities. In particular, sets of probabilities are just one possible instantiation of what might rightly be called 'imprecise probability', and the formalism of IPR provides a general abstract setting in which to explore the boundaries of this notion.

References

L.E.J. Brouwer. (1981). *Brouwer's Cambridge Lectures on Intuitionism*, Cambridge University Press.

L.E.J. Brouwer. *Intuitionism and Formalism*, Bulletin (New Series) of the American Mathematical Society, **37**, no. 1, 55–64.

H. Crane. (2017). The Logic of Probability and Conjecture. *Researchers.One*, https://www.researchers. 08-5.

H. Crane. *Intuitive Probabilistic Reasoning.* Book in progress.

H. Crane and I. Wilhelm. (2019). The Logic of Typicality. In *Statistical Mechanics and Scientific Explanation: Determinism, Indeterminism and Laws of Nature* (V. Allori, ed.). Available at *Researchers.One*, https://www.researchers.one/article/2018-08-18.

H. Curry and R. Feys. (1959). in Craig, William, ed., *Combinatory Logic, Volume I.*

B. de Finetti. (1937). La prévision: ses lois logiques, ses sources, subjectives. *Annales de l'Institut Henri Poincaré*, **7**, 1–68.

M. Dummett. (2000). *Elements of Intuitionism, 2nd edition.* Oxford Logic Guides (Book 39), Clarendon Press.

A. Heyting. (1971). Intuitionism: An introduction. *Study in Logic & Mathematics.*

W.A. Howard. (1969). The formulae-as-types notion of construction, in Seldin, Jonathan P., Hindley, J. Roger, to H.B. Curry: *Essays on Combinatory Logic, Lambda Calculus and Formalism*, 479–490.

C. Kapulkin and P.L. Lumsdaine. (2011). The Simplicial Model of Univalent Foundations (after Voevodsky). arXiv:1211.2851.

P. Martin-Löf. (1984). *Intuitionistic type theory*, Studies in Proof Theory. Lecture Notes, v. 1.

P. Martin-Löf. (1987). Truth of a Proposition, Evidence of a Judgment, Validity of a Proof. *Synthese*, **73**, 407–420.

P. Martin-Löf. (1996). On the Meanings of the Logical Constants and the Justifications of the Logical Laws. *Nordic Journal of Philosophical Logic*, **1**, 11–60.

G. Shafer. (1976). *A mathematical theory of evidence.* Princeton University Press.

D. Tsementzis. (2017). A meaning explanation for HoTT. *Synthese*.

Univalent Foundations Program. (2013). *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study.

P. Walley. (1990) . *Statistical Reasoning with Imprecise Probabilities.* Chapman & Hall/CRC Monographs on Statistics & Applied Probability, Volume 42.